MTH 303
Real analysis **Homework 4**

1. Let $\{a_n\}$ be a sequence of non-negative real numbers. Assume that the series $\sum_{n=1}^{\infty} a_n$ converges and the sum is s. Then show that $s = \sup\{s_n : n \in \mathbb{N}\}$, where s_n denotes the nth partial sum of the series.

2. Let $\{a_n\}$ and $\{b_n\}$ be two sequences such that $a_n = b_{n+1} - b_n$. Show that the series $\sum a_n$ converges if and only if $\lim_{n\to\infty} b_n$ exists.

3. Let $\{a_n\}$ and $\{b_n\}$ be two sequences.
   (a) If $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = l > 0$, then show that either both series $\sum a_n$ and $\sum_n b_n$ converge or both diverge.
   (b) Assume that $a_n > 0,\ b_n > 0$ and $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n},\ \forall n$. Show that if $\sum_n b_n$ converges then $\sum_n a_n$ converges.
   (c) Let $a_n \geq 0$ and $b_n \geq 0$. If series $\sum_n a_n$ and $\sum_n b_n$ converge, then show that the series $\sum \sqrt{a_n b_n}$ converges.

4. Let $\{a_n\}$ be a monotonically decreasing sequence of non-negative real numbers such that $\lim_{n\to\infty} a_n = 0$. If $\sum_n a_n$ is convergent then show that $\lim_{n\to\infty} n a_n = 0$.

5. Let $\sum_n a_n$ be a convergent series of non-negative real numbers. Show that
   (a) $\sum_n a_n^2$ is convergent.
   (b) $\sum_n \frac{\sqrt{a_n}}{n}$ is convergent.

6. If $\sum_n a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum_n a_n b_n$ converges.

7. Let $\{a_n\}$ be a monotonically decreasing sequence such that $\lim_{n\to\infty} a_n = 0$. Show that $\sum_n a_n \sin nx$ converges for all $x \in \mathbb{R}$ and $\sum_n a_n \cos nx$ converges for all $x \in \mathbb{R} \setminus \{2n\pi : n \in \mathbb{Z}\}$.

8. Prove that the Cauchy product of two absolutely convergent series is absolutely convergent.

9. Find the radius of convergence of each of the following power series
   (a) $\sum_{n=1}^{\infty} n^3 z^n$
   (b) $\sum_{n=1}^{\infty} \frac{2^n}{n!} z^n$
   (c) $\sum_{n=1}^{\infty} \frac{2^n}{n^2} z^n$
   (d) $\sum_{n=1}^{\infty} \frac{n^3}{3^n} z^n$

10. Suppose that the coefficients of the power series $\sum_{n=1}^{\infty} c_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

11. If $\sum_n a_n$ is a conditionally convergent series, then prove that the series of its positive terms and the series of its negative terms are both divergent.

12. Consider the alternating series $\sum_n \frac{(-1)^{n+1}}{n}$ and one of its rearrangement

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots$$

in which two positive terms are always followed by one negative term. Let s be the sum of the original series, then show that the rearranged series converges to a number different from s.